Problem Set 4

Macroeconomics III

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Problem 1

Households live for two periods. In each period, the households can divide their time between leisure $x_t \in [0,1]$ and working $1 - x_t$. Hence, their labour income is $w_t(1 - x_t)$ in period t. The households'

maximization problem is, therefore:

$$\max_{\{c_0, c_1, x_0, x_1\}} \quad \ln(c_0) + \frac{x_0^{1-\varphi}}{1-\varphi} + \beta \left[\ln c_1 + \frac{x_1^{1-\varphi}}{1-\varphi} \right], \quad \varphi > 0, \varphi \neq 1$$

$$\text{s.t.} \quad c_0 + \frac{c_1}{R_1} + w_0 x_0 + \frac{w_1 x_1}{R_1} = w_0 + \frac{w_1}{R_1}$$

The right hand side of the budget constraint (BC) is the lifetime income if the households choose to work all the time. The left hand side is the lifetime consumption and the opportunity cost of leisure. Hence, leisure can be thought of as a good with price w_t.

The lifetime BC is derived by combining the BCs:

$$c_0 + s = (1 - x_0) w_0 \qquad \text{and} \qquad c_1 = (1 - x_1) w_1 + s R_1$$

Problem 1a - FOCs

Find first order conditions:

We set-up the Lagrangian:

$$\mathcal{L} = \ln(c_0) + \frac{x_0^{1-\varphi}}{1-\varphi} + \beta\left[\ln c_1 + \frac{x_1^{1-\varphi}}{1-\varphi}\right] + \lambda\left(w_0 + \frac{w_1}{R_1} - c_0 - \frac{c_1}{R_1} - w_0 x_0 - \frac{w_1 x_1}{R_1}\right)$$

We find first order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_0} = 0 \quad &\Longrightarrow \quad \frac{1}{c_0} = \lambda \\
\frac{\partial \mathcal{L}}{\partial c_1} = 0 \quad &\Longrightarrow \quad \frac{\beta}{c_1} = \frac{\lambda}{R_1} \\
\frac{\partial \mathcal{L}}{\partial x_0} = 0 \quad &\Longrightarrow \quad x_0^{-\varphi} = \lambda w_0 \\
\frac{\partial \mathcal{L}}{\partial x_1} = 0 \quad &\Longrightarrow \quad \beta x_1^{-\varphi} = \lambda \frac{w_1}{R_1}
\end{aligned}$$

Problem 1b - Intertemporal choice of leisure

Combine the first order conditions to characterize the inter temporal choice of leisure. Show how higher values of φ lead the household to choose a smoother leisure path over time. Interpret.

We combine the last two equations:

$$\frac{\beta R_1 x_1^{-\varphi}}{w_1} = \frac{x_0^{-\varphi}}{w_0} \quad \Longrightarrow \quad \frac{x_1}{x_0} = \left(\beta R_1 \frac{w_0}{w_1}\right)^{\frac{1}{\varphi}}$$

This equation looks a lot like the Euler equation from the Ramsey model and the intuition is very similar but with leisure instead of consumption. $\frac{1}{\varphi}$ is the elasticity of intertemporal substitution of leisure.

For a very high φ agents will choose to smooth leisure so they work the same amount each period (when $\varphi \to \infty$ then $\frac{1}{\varphi} \to 0$ and $\frac{x_1}{x_0} \to 1$). The risk-aversion interpretation is the same as in the previous problem sets but where they are risk averse wrt. the amount of leisure.

Problem 2

Household problem

Households live for two periods: young and old. They work and earn a wage w_t when young. They live off their savings when old.

Households maximizes

$$U = \ln(c_{1t}) + \beta \ln(c_{2t+1}), \quad \beta \in (0,1)$$

They consume c_{1t} when young and c_{2t+1} when old.

The budget constraints when young and old are:

$$\text{BC when young:} \qquad c_{1t} + s_t = w_t$$
$$\text{BC when old:} \qquad c_{2t+1} = (1 + r_{t+1})s_t$$

The maximization problem is therefore:

$$\max_{c_{1t},\ c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$
$$\text{s.t.} \quad c_{1t} = w_t - s_t$$
$$c_{2t+1} = (1 + r_{t+1})s_t$$

Problem 2 - Savings rate

We want to find the optimal s_t. We insert the budget constraints into the utility function:

$$\max_{s_t} \quad \ln(w_t - s_t) + \beta \ln\left[(1 + r_{t+1})s_t\right]$$

The maximization problem is now a function of one single variable s_t. We find the first order condition wrt. s_t and isolate s_t:

$$\begin{aligned} \frac{\beta}{s_t} &= \frac{1}{w_t - s_t} \\ s_t &= \beta(w_t - s_t) \\ s_t(1+\beta) &= \beta w_t \qquad \Longrightarrow \qquad s_t = \frac{\beta}{1+\beta} w_t \end{aligned}$$

The savings rate is independent of the interest rate since we have assumed log-utility as the substitution effect offsets the income effect.

Substitution effect: When r_t decreases, the return to savings decrease, which makes it less attractive

Income effect: When r_t decreases, the value of savings decrease, so HH have to save more to maintain a constant consumption level.

Problem 2 - Capital accumulation

Since we have Cobb-Douglas production, the wage rate and interest rate will be:

$$r_t = A\alpha k_t^{\alpha-1}$$
$$w_t = A(1-\alpha)k_t^{\alpha}$$

We insert this into the equation for s_t:

$$s_t = \frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}$$

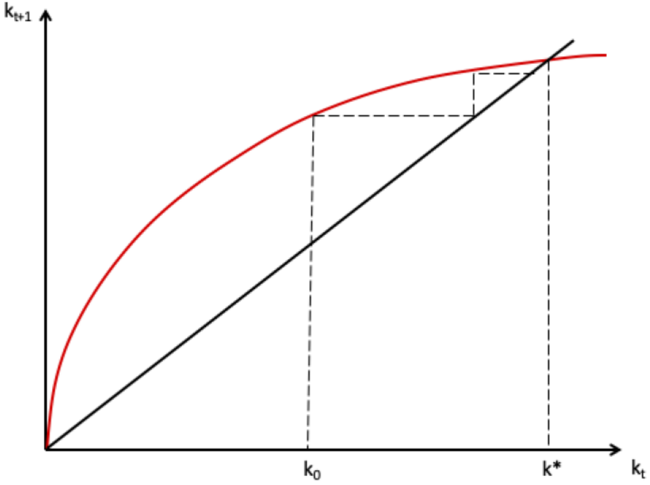
The savings are used as capital in production in the following period:

$$K_{t+1} = S_t = s_t L_t$$
$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t s_t}{(1+n)L_t} = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}$$

This is the equation for capital accumulation. Next we look at how different shocks affects the capital accumulation.

Problem 2 - Capital accumulation graphically

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha} \tag{1}$$



Problem 2a - Increase in population growth

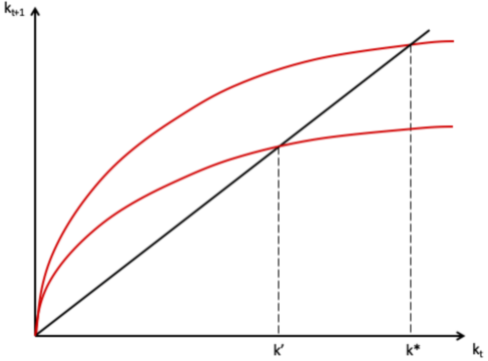
$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}$$

As n increases, the capital accumulation schedule shifts downward.

$$n \uparrow \implies k_{t+1} \downarrow \implies w_{t+1} \downarrow \implies s_{t+1} \downarrow$$

Dynamics:

- Capital is shared by more workers.
- Capital per worker decreases.
- MPL decreases so the wage rate decreases.
- Savings decrease as the wage decreases.
- Capital per worker decreases.



Problem 2b - Decrease in A

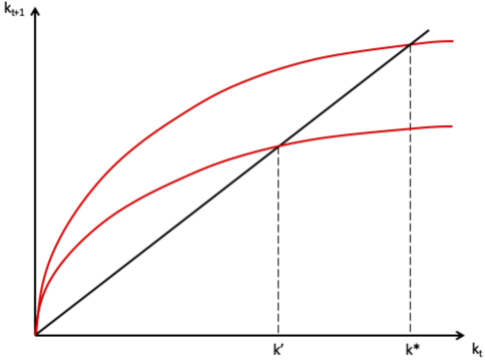
$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}$$

From equation 1, we see that the effect is equal to an increase in n.

$$A \downarrow \implies \text{MPL} \downarrow \implies w_t \downarrow \implies s_t \downarrow$$

Dynamics:

- Technology decreases productivity decreases.
- Wage rate decreases, which lowers savings.
- As savings decrease, capital decrease.
- As capital decreases, MPL and wages decrease.



Problem 2c - $\alpha \uparrow$

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}$$

Two effects from $\alpha \uparrow$:

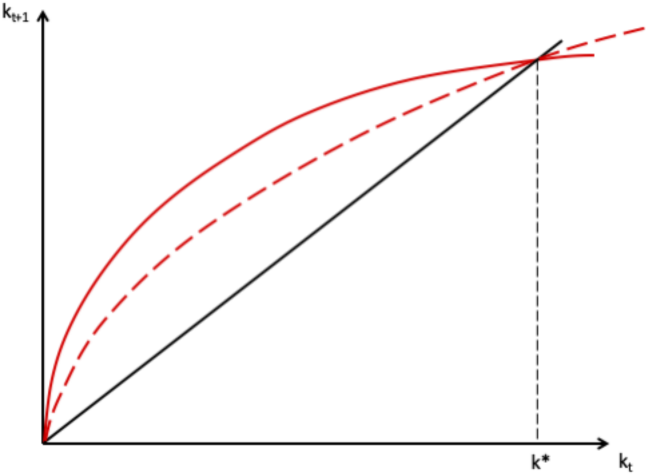
Effect 1: Capital more productive, which increases wages, which leads to higher savings.

Effect 2: The income share of labour decreases, which lowers wages, all else equal, which will lower savings.

$$w_t = A(1 - \underbrace{\alpha}_{\text{Effect 2}}) \underbrace{k_t^{\alpha}}_{\text{Effect 1}}$$

The overall effect from $\alpha \uparrow$ on capital accumulation is ambiguous. The shape of the function changes as the shape gets closer to a linear line.

Problem 2c - $\alpha \uparrow$ - graphically



This is the case where effect 1 and 2 perfectly offsets each other.

Problem 3

Introduce depreciation in the Diamond model such that $r_t = f'(k_t) - \delta$, where δ is the rate at which capital depreciates.

a) Does this affect the capital accumulation schedule $k_{t+1}(k_t)$? How?

b) For the case log CD and $\delta = 1$, what is the capital accumulation schedule $k_{t+1}(k_t)$?

Solution a: Since the savings decision does not depend on the interest rate, the depreciation rate does not affect the capital accumulation. This is because we have assumed log-utility, so the substitution effect and the income effect perfectly cancel each other.

Furthermore, since the old households consume all their savings in each period, capital accumulation is not affected.

Solution b: We assumed Cobb-Douglas production and log-utility problem 2, so the capital accumulation is given by equation 1.

Problem 4

Consider an OLG economy where individuals live for two periods.

- Constant population and homogeneity within cohorts.
- Endowment when young: e_t
- Endowment when old: $(1+g)e_t$ Where g can be positive or negative.
- Return on investments/savings: $1+r$
- Endowment grows according to: $e_{t+1} = (1+m)e_t$

Each individual maximizes:

$$U_t = \ln(c_{1t}) + \frac{1}{1+\rho}\ln(c_{2t+1})$$

Note that the discount factor is written as $\frac{1}{1+\rho}$ but has the same interpretation as β. As far as I know, people usually use $\frac{1}{1+\rho}$ in the OLG framework and β in the Ramsey framework.

Problem 4a - Optimal savings rate (1/2)

The household problem is:

$$\begin{aligned}\max_{c_{1t}, c_{2t+1}} \; & \ln(c_{1t}) + \frac{1}{1+\rho}\ln(c_{2t+1}) \\ \text{s.t.} \quad & c_{1t} = e_t - s_t \\ & c_{2t+1} = (1+r)s_t + (1+g)e_t\end{aligned}$$

Substituting the budget constraints into the utility function leads to[1]:

$$\max_{s_t} \ln(e_t - s_t) + \frac{1}{1+\rho}\ln\left[(1+r)s_t + (1+g)e_t\right]$$

The first order condition is then:

$$-\frac{1}{e_t - s_t} + \frac{1+r}{1+\rho}\left(\frac{1}{(1+r)s_t + (1+g)e_t}\right) = 0$$

[1] The problem can also be solved by finding FOCs wrt. to c_{1t} and c_{2t+1}.

Problem 4a - Optimal savings rate (2/2)

$$-\frac{1}{e_t - s_t} + \frac{1+r}{1+\rho}\left(\frac{1}{(1+r)s_t + (1+g)e_t}\right) = 0$$

We isolate s_t in the FOC.

$$\begin{aligned}
\frac{1}{e_t - s_t} &= \frac{1+r}{1+\rho}\left(\frac{1}{(1+r)s_t + (1+g)e_t}\right) \\
e_t - s_t &= \frac{1+\rho}{1+r}\Big((1+r)s_t + (1+g)e_t\Big) \\
e_t - s_t &= (1+\rho)s_t + e_t\left(\frac{(1+g)(1+\rho)}{1+r}\right) \\
(2+\rho)s_t &= e_t\left(1 - \frac{(1+g)(1+\rho)}{1+r}\right) \\
s_t &= e_t\frac{1}{2+\rho}\left(\frac{1+r-(1+g)(1+\rho)}{1+r}\right) \\
s_t &= e_t\left(\frac{1+r-(1+g)(1+\rho)}{(1+r)(2+\rho)}\right)
\end{aligned}$$

Problem 4a - Effect of an increase in g on $\hat{s}_t$

What is the effect of an increase in g on the individual saving rate?

The savings rate for young people is savings divided by the endowment:

$$\hat{s} \equiv \frac{s_t}{e_t} = \frac{1 + r - (1+g)(1+\rho)}{(1+r)(2+\rho)}$$

It is clear from $\hat{s}_t$ that $g \uparrow \implies \hat{s}_t \downarrow$.

When the endowment while old grows, the young will choose to save less in order to smooth consumption. This is an income effect. This can also be shown by calculating the derivative wrt. g:

$$\frac{\partial \hat{s}}{\partial g} = -\frac{1+\rho}{(1+r)(2+\rho)} < 0$$

Note: We don't know whether $\hat{s}$ is positive or negative since it depends on the parameters. Neither do we know if savings are allowed to be negative (loans).

Problem 4b - Effect of an increase in m on S_t

What is the effect of an increase in m on aggregate net saving?

The aggregate savings is given by, $S_t = Ls_t$.[2] Aggregate net savings is the difference in aggregate savings from period t to $t+1$, $S_{t+1} - S_t$:

$$\begin{aligned}\Delta S_{t+1} = S_{t+1} - S_t = L(s_{t+1} - s_t) &= L(\hat{s}e_{t+1} - \hat{s}e_t) \\ &= L\hat{s}\big(e_t(1+m) - e_t\big) = L\hat{s}me_t\end{aligned}$$

The derivative of the change wrt. m is:

$$\frac{\partial L\hat{s}me_t}{\partial m} = L\hat{s}e_t$$

As m increases, individuals will receive a larger endowment both as young and as old. The change in the aggregate savings will have the same sign as $\hat{s}$.

The savings fraction of the endowment is unchanged as m increases.

[2] There is no population growth so $L_t = L_{t+1} = L$.

Problem 4c - Effect of increase in m when $g = m$ (1/2)

Assume $m = g$. What is the effect of an increase in m on aggregate net saving?

The savings rate is now:

$$\hat{s} = \frac{1 + r - (1+m)(1+\rho)}{(1+r)(2+\rho)}$$

The derivative wrt. m of the savings rate is:

$$\frac{\partial \hat{s}}{\partial m} = -\frac{1+\rho}{(1+r)(2+\rho)} < 0$$

The derivative of the aggregate savings rate is then:

$$\frac{\partial (S_{t+1} - S_t)}{\partial m} = \frac{\partial L \hat{s} m e_t}{\partial m} = \underbrace{L \hat{s} e_t}_{\lesseqgtr 0} + \underbrace{L \frac{\partial \hat{s}}{\partial m} m e_t}_{<0}$$

We see that there are two diverging effects.

Problem 4c - Effect of increase in m when $g = m$ (2/2)

The overall effect of an increase in m is ambiguous:

$$\frac{\partial(S_{t+1} - S_t)}{\partial m} = \underbrace{L\hat{s}e_t}_{\lesseqgtr 0} + \underbrace{L\frac{\partial \hat{s}}{\partial m}me_t}_{<0}$$

Two effects:

- The first effect $L\hat{s}e_t$ is positive if $\hat{s} > 0$. This is the effect if households did not change the savings fraction. It can be seen as a mechanic effect. It is the effect from problem 4b.
- The second effect $L\frac{\partial \hat{s}}{\partial m}me_t$ is negative. This effect is based on the change in savings behaviour. As m increases, the individuals receive a larger endowment when old, which leads them to save relatively less. This is the effect from problem 4a.

The overall effect is ambiguous.

Problem 4d - High savings rate $\implies$ high output growth?

Based on your answers analyze the theoretical validity of the following statement: "In China, the high growth rate of output is the result of a high saving rate"

The growth rate is exogenous in our model. Hence, this model is not appropriate for evaluating what causes output growth.

Furthermore, the output growth affects the savings decision in our model and not the opposite. Therefore, our model suggests that the causality is in fact the reverse.

Additionally, the relation between growth and savings is ambiguous, as it is negatively correlated in problem a), positively correlated in problem b) and ambiguous in problem c).

Our model does not support the statement (and it is not really appropriate to evaluate the statement based on this model).